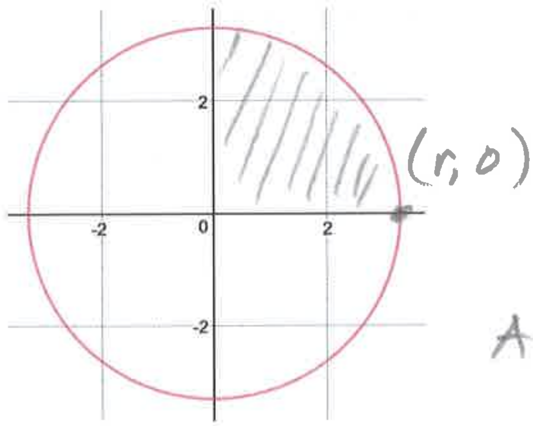
Trigonometric Substitution

Example 1:

Prove that the area of a circle with radius r is πr^2.



$$x^2+y^2=r^2$$

$$y=\sqrt{r^2-x^2}$$

$$A = 4\int_0^r \sqrt{r^2-x^2}\,dx$$

Let $x = r\sin\theta$, $x^2 = r^2\sin^2\theta$, $\sqrt{r^2-x^2} = r\cos\theta$

$\theta = \sin^{-1}\frac{x}{r}$. when $x=0$, $\theta = 0$.

when $x=r$, $\theta = \frac{\pi}{2}$

$$dx = r\cos\theta\,d\theta$$

$$A = 4\int_0^{\frac{\pi}{2}} r^2\cos^2\theta\,d\theta = 4r^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= 4r^2\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos2\theta}{2}\right)dx = 4r^2\left(\frac{\theta}{2}+\frac{\sin2\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$

$$= 4r^2\left(\frac{\pi}{4}+0-0\right) = \pi r^2$$

Example 2:

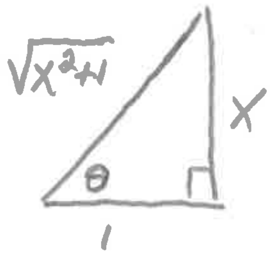
Evaluate $\int \frac{1}{x^2\sqrt{x^2+1}}dx$.

Radicals are troublesome

so let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$\theta = \tan^{-1} x$



$$\int \frac{\sec^2\theta}{\tan^2\theta(\sec\theta)} d\theta = \int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta = \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

Let $u = \sin\theta$, $du = \cos\theta \, d\theta$.

$$\int \frac{1}{u^2} du = -\frac{1}{u} + C$$

$$= \frac{-1}{\sin\theta} + C = \frac{-1}{\frac{x}{\sqrt{x^2+1}}} + C$$

$$= \frac{-\sqrt{x^2+1}}{x} + C .$$

Example 3:

Evaluate $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$.

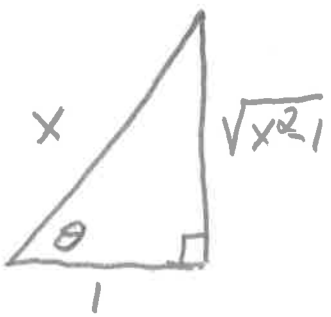
Radicals are a pain.

Let $x = \sec\theta$,

$dx = \sec\theta \tan\theta$

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$\theta = \sec^{-1} x$,



$$\int \frac{\sec\theta \tan\theta}{\sec^2\theta \tan\theta}\,d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta$$

$$= \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$$