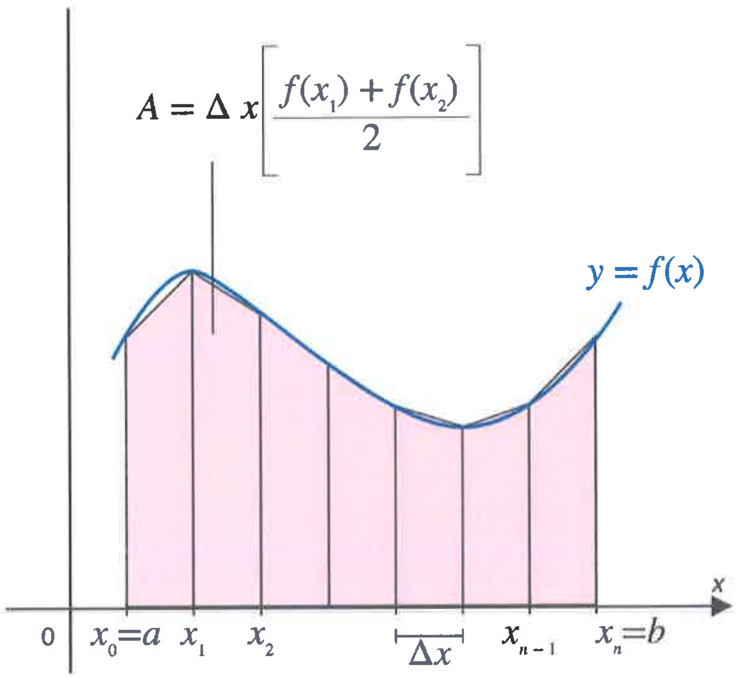
Numerical Integration, The Trapezoid Rule



An approximation of $\int_a^b f(x)dx$ using the Trapezoidal Rule is the sum

$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)],$$

where $\Delta x = \dfrac{(b-a)}{n}$ and $x_i = a + i\Delta x$ for $i = 0, 1, \ldots, n$.

Theorem: Error Estimate for the Trapezoidal Rule

If f'' is continuous on $[a, b]$ and M is an upper bound for $|f''(x)|$ then the error E_T between the exact value of $\int_a^b f(x)dx$ and the Trapezoidal Rule approximation T_n satisfies

$$|E_T| \leq \frac{M(b-a)^3}{12n^2},$$

where n is the number of subintervals in the partition.

Example:

Use the Trapezoidal Rule with $n = 8$ to approximate the integral $\int_0^8 2x^4 dx$. Round any intermediate calculations, if needed, to no less than six decimal places, and round your final answer to four decimal places.

$$T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$\Delta x = \frac{8-0}{8} = 1$. $\quad f(x) = 2x^4$, $2f(x) = 4x^4$

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, ..., $x_8 = 8$

$f(x_0) = 0$, $2f(x_1) = 4$, $2f(x_2) = 4(2)^4$, $2f(x_3) = 4\cdot(3)^4$

$2f(x_4) = 4(4)^4$, $2f(x_5) = 4(5)^4$, $2f(x_6) = 4(6)^4$

$2f(x_7) = 4(7)^4$, $f(8) = 2(8)^4$. So $T_8 = 13{,}448$

$\int_0^8 2x^4 dx = \left.\frac{2x^5}{5}\right|_0^8 = 13{,}107.2$

$f'(x) = 8x^3$, $f''(x) = 24x^2$, $|f''(x)| \leq 24(8^2)$

$= 1536$ on $[0,8]$. If we want the

error $< 10^{-3}$, we solve $\frac{1536(8)^3}{12n^2} < 10^{-3}$

which yields $n > 8095.4$. Choose

$n \geq 8096$ (even)