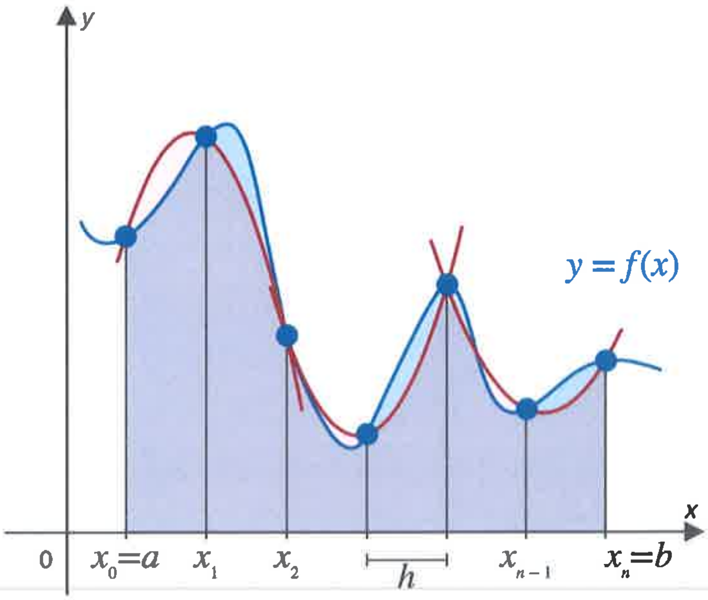
Numerical Integration, Simpson's Rule



An approximation of $\int_a^b f(x)dx$ using Simpson's Rule is the sum

$$S_n = \frac{\Delta x}{3}\left[\begin{array}{l} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{array}\right]$$

where n is even, $\Delta x = \dfrac{(b-a)}{n}$, and $x_i = a + i\Delta x$ for $i = 0, \ldots, n$. (Note the pattern of the coefficients: 1, 4, 2, 4, 2, …, 2, 4, 1.)

Theorem: Error Estimate for Simpson's Rule

If $f^{(4)}$ is continuous on $[a, b]$ and M is an upper bound for $|f^{(4)}(x)|$, then the error E_S between the exact value of $\int_a^b f(x)dx$ and the Simpson's Rule approximation S_n satisfies

$$|E_S| \leq \frac{M(b-a)^5}{180n^4},$$

where n is the number of subintervals in the partition.

Example:

Use Simpson's Rule with $n = 8$ to approximate the integral $\int_0^8 2x^4 dx$. Round any intermediate calculations, if needed, to no less than six decimal places, and round your final answer to four decimal places.

$$S_n = \frac{\Delta x}{3}\begin{bmatrix} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots \\ +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \end{bmatrix}$$

$\Delta x = \frac{8-0}{8} = 1$, $f(x) = 2x^4$, $2f(x) = 4x^4$, $4f(x) = 8x^4$

$x_0 = 0, x_1 = 1, x_2 = 2, \ldots, x_8 = 8$

$f(x_0) = 0$, $4f(x_1) = 8$, $2f(x_2) = 4(2)^4$

$4f(x_3) = 8(3)^4$, $2f(x_4) = 4(4)^4$,

$4f(x_5) = 8(5)^4$, $2f(x_6) = 4(6)^4$, $4f(x_7) = 8(7)^4$

$f(x_8) = 2(8)^4$. $S_8 = 13{,}109.33$

$\int_0^8 2x^4 dx = \frac{2x^5}{5}\Big|_0^8 = 13{,}107.2$

$f'(x) = 8x^3$, $f''(x) = 24x^2$, $f^{(3)}(x) = 48x$, $f^{(4)}(x) = 48$

If we want the error $< 10^{-3}$, we solve

$\frac{48(8)^5}{180n^4} < 10^{-3}$ which yields $n > 54.36$

choose $n \geq 56$ (even)