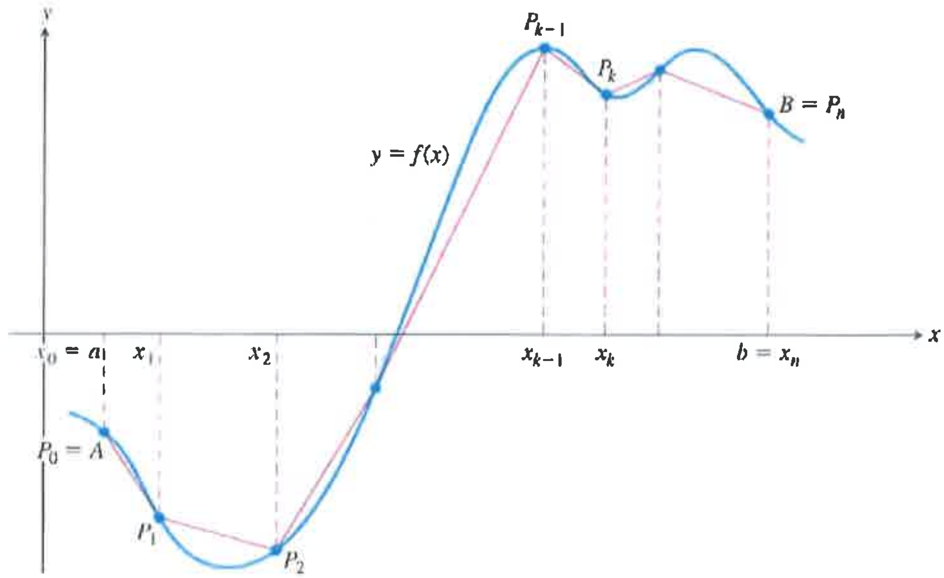
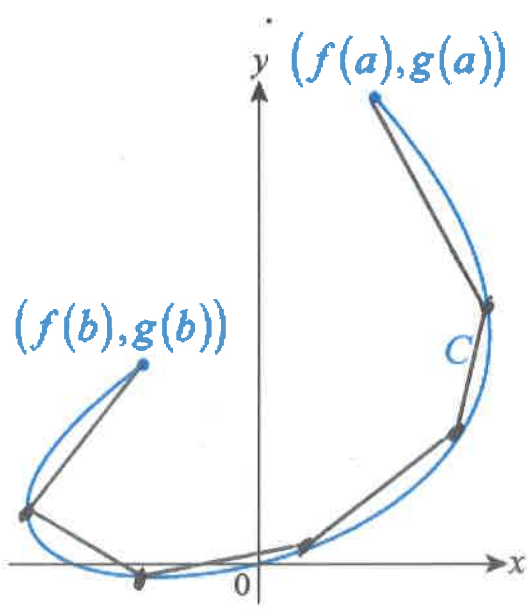
Parametric Equations and Arc Length

Suppose $f: [a, b] \rightarrow R$ and f' is continuous on $[a,b]$.



$$L = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$



For a smooth curve *C* defined parametrically by *x = f(t)* and *y = g(t)* over the interval [*a*,*b*] the formula for arc length becomes

$$L = \int_{(f(a),g(a))}^{(f(b),g(b))} ds = \int_{(f(a),g(a))}^{(f(b),g(b))} \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example:

Find the length of the curve defined by $x = \sqrt{t}$, $y = \frac{t^2}{16} + \frac{1}{2t}$ over the interval $7 \le t \le 9$.

$$x' = \frac{1}{2\sqrt{t}}, \quad y' = \frac{2t}{16} + \frac{1}{2}\left(\frac{-1}{t^2}\right) = \frac{t}{8} - \frac{1}{2t^2}$$

$$(x')^2 = \frac{1}{4t}, \quad (y')^2 = \left(\frac{2t^3-8}{16t^2}\right)^2 = \left(\frac{t^3-4}{8t^2}\right)^2$$

$$(x')^2 + (y')^2 = \frac{1}{4t} + \frac{t^6 - 8t^3 + 16}{64t^4}$$

$$= \frac{16t^3 + t^6 - 8t^3 + 16}{64t^4}$$

$$= \frac{t^6 + 8t^3 + 16}{64t^4} = \frac{(t^3+4)^2}{(8t^2)^2}$$

$$\int_7^9 \sqrt{\frac{(t^3+4)^2}{(8t^2)^2}}\, dt = \int_7^9 \frac{t^3+4}{8t^2}\, dt = \int_7^9 \left(\frac{t}{8} + \frac{1}{2t^2}\right) dt$$

$$= \left(\frac{t^2}{16} - \frac{1}{2t}\right)\Big|_7^9 = \frac{81}{16} - \frac{1}{18} - \left(\frac{49}{16} - \frac{1}{14}\right) = \frac{127}{63}$$