Infinite Series, An Introduction

Suppose the function f is given by $f(x) = e^x\ , -\infty < x < \infty$.

What is the value of $f(2) = e^2$?

Easy answer, your calculator yields $e^2 \cong 7.389056$.

But how did it do it? Does it have every value of the function in storage? Of course not.

It is pretty clear how a calculator would a simple program to evaluate a polynomial at some specific value of x .

Consider the polynomial given by $p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$.

n	$p_n(x)$	$p_n(2)$
2	$1 + x + \frac{x^2}{2!}$	5
3	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$	6.3333
4	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$	7
5	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^5}{5!}$	7.2667
6	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^6}{6!}$	7.3556
7	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^7}{7!}$	7.3809
8	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^8}{8!}$	7.3872
9	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^9}{9!}$	7.3886
10	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$	7.3889
11	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{11}}{11!}$	7.8897
12	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{12}}{12!}$	7.3889
		$e^2 \cong 7.389056$

We can write: $$e^2 = \lim_{n \to \infty} p_n(2)$$

$$e^2 = \lim_{n \to \infty} \left(1 + 2 + \frac{2^2}{2!} + \cdots + \frac{2^n}{n!} \right)$$

$$e^2 = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{2^k}{k!}$$

$$e^x = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{x^k}{k!} , \quad \infty < x < \infty$$