Monotonic Functions and the 1st Derivative Test

Four important consequences of the Mean Value Theorem:

1. If $f'(x) > 0$ for each $x$ in $(a, b)$ then $f$ is (strictly) increasing on $(a, b)$.

2. If $f'(x) < 0$ for each $x$ in $(a, b)$ then $f$ is (strictly) decreasing on $(a, b)$.

3. If $f'(x) = 0$ for each $x$ in $(a, b)$ then $f$ is a constant function on $(a, b)$.

4. If $f'(x) = g'(x)$ for each $x$ in $(a, b)$ then there exists a constant $c$ such that $f(x) = g(x) + c$ for each $x$ in $(a, b)$.

Example

Find the critical points of the function $f$ defined by $f(x) = x^3 - 27x - 4$ and determine the intervals on which $f$ is increasing and the intervals on which $f$ is decreasing. Can you determine the local extreme values from this analysis?
First Derivative Test:

Suppose $x = c$ is a critical point for $f$.

If $f'$ changes from negative to positive at $c$ then $f$ has a local minimum at $c$.

If $f'$ changes from negative to positive at $c$ then $f$ has a local maximum at $c$.

If $f'$ does not change sign at $c$ then $f$ has no local extremum at $c$. 
Example

Let $f$ be defined by $f(x) = e^{2x} + e^{-x}$. Find the local extreme values and sketch the graph.
Example

Let $w(x) = x^2 (x^2 - 4)$. Find the intervals on which $w$ is increasing and the intervals on which $w$ is decreasing. Find the local extreme values and sketch the graph.
Concavity and Curve Sketching

Example

Consider the function $f$ defined by $f(x) = x^3$. Note that

\[ f'(x) = 3x^2, \quad f''(x) = 6x \]

\[ f''(x) > 0 \quad \text{for} \quad x > 0 \]

\[ f''(x) < 0 \quad \text{for} \quad x < 0 \]

$f'$ is increasing on $(0, \infty)$

$f'$ is decreasing on $(-\infty, 0)$

$f''$ changes sign at $x = 0$

\[ -5 \quad 0 \quad 5 \]

Definition

The function $f$ is concave up on the interval $(a,b)$ if $f'$ is increasing on $(a,b)$.

The function $f$ is concave down on the interval $(a,b)$ if $f'$ is decreasing on $(a,b)$. 
Definition

\((c, f(c))\) is a point of inflection if the graph of \(f\) has a tangent line at \((c, f(c))\) and the concavity of \(f\) changes at \((c, f(c))\).

Example

Let \(f(x) = x(6-2x)^2\). Identify all extreme values, the intervals where \(f\) is increasing or decreasing, where it is concave up, where it is concave down, its inflection points, and sketch the graph.
Second Derivative Test

Suppose \( f'' \) is continuous near \( x = c \).

1. If \( f'(c) = 0 \) and \( f''(c) < 0 \) then \( f \) has a local maximum at \( x = c \).

2. If \( f'(c) = 0 \) and \( f''(c) > 0 \) then \( f \) has a local minimum at \( x = c \).

3. If \( f'(c) = 0 \) and \( f''(c) = 0 \) then the test fails.
Sketch the graphs of the following:

a) \( f(x) = x + 5\sin x \)

b) \( g(x) = 2x - 3\frac{x^2}{3} \)
c) \( h(x) = \sqrt{|x-4|} \)

d) \( k(x) = \frac{4x}{x^2 + 4} \)
e) \( s(x) = \frac{x-1}{x^2(x-2)} \)