Vectors

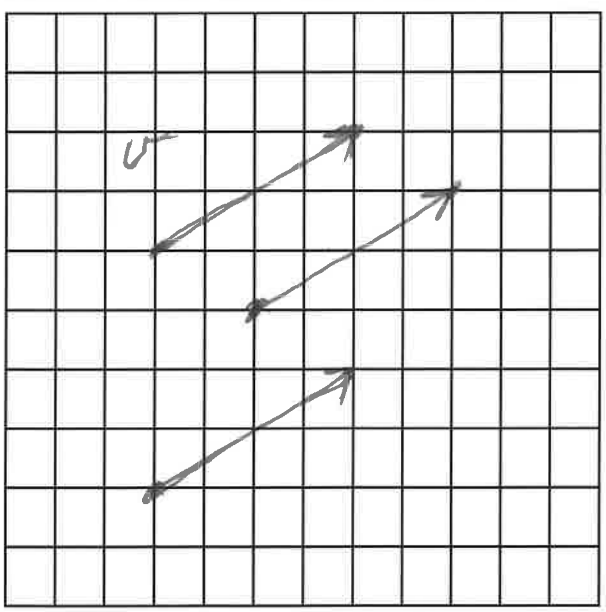
Many times in mathematics we represent quantities as real numbers. Other quantities cannot be represented by a single real number because the quantities involve both a magnitude and a direction. List some examples:

Rocket launch : thrust, direction

Gravitational attraction, plane in flight

Such quantities are called vectors and can be represented as directed line segments where the length of the line segment corresponds to the magnitude of the quantity and the direction of the line segment corresponds to the direction of the quantity. Directed line segments that have the same magnitude and same direction are representations of the same vector.

Draw three directed line segments on the grid below that are the same vector and label them **v.**



Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, the components for a vector **a** that is the directed line segment from A to B (denoted $\overrightarrow{AB}$) is given by:

$\mathbf{a} = \overrightarrow{AB} = < x_2 - x_1, y_2 - y_1 >$

Find the components of the vector represented by the line segment from the point $(-3, \frac{2}{3})$ to the point $(-\frac{1}{2}, -4)$

$< -\frac{1}{2} + 3, -4 - \frac{2}{3} >$

$< \frac{5}{2}, \frac{-14}{3} >$

The length (or magnitude) of the vector $\mathbf{a} = < a_1, a_2 >$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$

Find the magnitude of the vector represented by the line segment from the point $(-3, \frac{2}{3})$ to the point $(-\frac{1}{2}, -4)$

$a = < \frac{5}{2}, -\frac{14}{3} >$

$|a| = \sqrt{\frac{25}{4} + \frac{196}{9}} = 5.2941$

Vector Operations:
Let $\mathbf{a} = < a_1, a_2 >$ and let $\mathbf{b} = < b_1, b_2 >$ and let c be any real number.

$\mathbf{a} + \mathbf{b} = < a_1 + b_1, a_2 + b_2 >$
$c\mathbf{a} = < ca_1, ca_2 >$

Example:
Let $\mathbf{a} = < 3, 4 >$ and $\mathbf{b} = < -2, -2 >$

Find the components of the vector
$\mathbf{a} + \mathbf{b}$

$= < 1, 2 >$

$-2\mathbf{a} + 3\mathbf{b} \quad = < -6, -8 > + < -6, -6 >$

$= < -12, -14 >$

Definition:
A **unit vector** is a vector that has a magnitude of 1.

What are some examples of a unit vectors?

$\langle 0, 1 \rangle$, $\langle 1, 0 \rangle$, $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\langle 0, -1 \rangle$, $\langle -1, 0 \rangle$, $\langle \frac{3}{5}, -\frac{4}{5} \rangle$

Find the component form of the vector **a** if **a** is a unit vector in the direction of $< 2, \, - 3 >$.

$$\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \rangle$$

Find the component form of the vector **v** if $||\mathbf{v}|| = 13$ and **v** is in the direction of $< 2, \, - 3 >$.

$\vec{v} = \langle 2a, -3a \rangle$, $\|\vec{v}\| = 13$

so $\sqrt{4a^2 + 9a^2} = 13$

$4a^2 + 9a^2 = 169$

$13a^2 = 169$, $a^2 = 13$

$a = \sqrt{13}$, $\vec{v} = \langle 2\sqrt{13}, -3\sqrt{13} \rangle$

Standard Basis Vectors

$\mathbf{i} = \ < 1, 0 >$

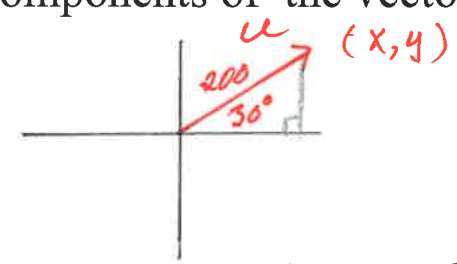
$\mathbf{j} = \ < 0, 1 >$

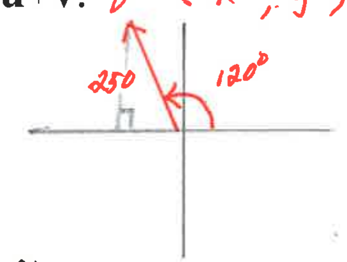
Representing a vector as a sum of the standard basis vectors:

$< a_1, a_2 > \ = a_1\mathbf{i} + a_2\mathbf{j}$

Example:

The vector **u** has a magnitude of 200 and forms a 30° angle with the positive x-axis. The vector **v** has a magnitude of 250 and forms a 120° angle with the positive x-axis. Find the components of vectors **u** and **v** and then find the components of the vector **u**+**v**. $v \ (x, y)$





$\sin 120 = \frac{y}{250}$

$y = 125\sqrt{3}$

$\cos 120 = \frac{x}{250}$

$-\frac{1}{2} = \frac{x}{250}$

$x = -125$

$\sin 30 = \frac{1}{2} = \frac{y}{200}, \quad y = 100$

$\cos 30 = \frac{\sqrt{3}}{2} = \frac{x}{200}, \quad x = 100\sqrt{3}$

$v = \langle -125, 125\sqrt{3} \rangle$

$u = \langle 100\sqrt{3}, 100 \rangle$

$u + v = \langle 100\sqrt{3} - 125, 100 + 125\sqrt{3} \rangle$

Angles and Components:

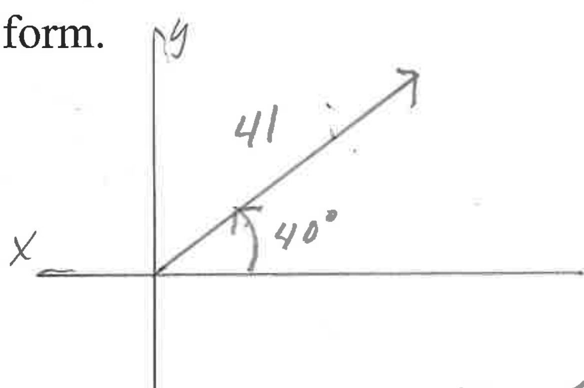
The two-dimensional vector **v** with magnitude A that makes an angle of θ with the positive x-axis is given by:

$$\mathbf{v} = (A\,cos\,\theta)\,\mathbf{i} + (A\,sin\,\theta)\,\mathbf{j} = \ < A\,cos\,\theta, A\,sin\,\theta >$$

The acute angle θ formed by the vector $\mathbf{v} = \ < v_1, v_2 >$ and the x-axis is given by:

$$\theta = tan^{-1}\left(\frac{|v_2|}{|v_1|}\right)$$

A ball was hit into the air at a speed of 41 mph at an angle of 40^o from the horizontal. Express this velocity in vector form.



$v = < a, b >$

$|v| = 41$

$a = 41 \cos 40^\circ$

$b = 41 \sin 40^\circ$

$v = < 31.409, 26.354 >$