The Fundamental Theorem of Algebra

Theorem:
If $f(x)$ is a polynomial function of degree n with leading term $a_n x^n$, then f has precisely n linear factors of the form.
$f(x) = a_n(x - c_1)(x - c_2)...(x - c_n)$
where $c_1, c_2, \ldots, c_n$ are complex numbers (not necessarily unique).

some or all could be real numbers

What does this theorem say about the polynomial
$f(x) = x^3 + 2x^2 - 2x - 4$?
Hint: -2, $\sqrt{2}$, $-\sqrt{2}$ are the only zeros of f.

$f(x) = (x+2)(x-\sqrt{2})(x+\sqrt{2})$

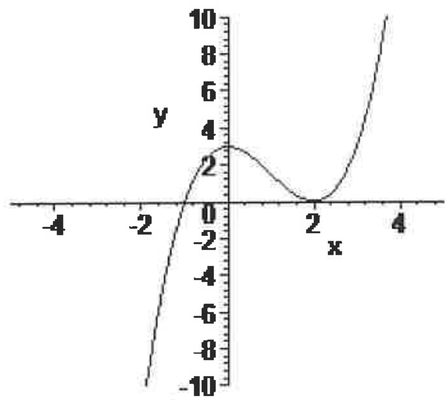
Definition:
If $(x - c)^n$ is a factor of a polynomial, then c is said to be a zero of the polynomial with **multiplicity n.** *To determine the multiplicity of a zero of a function you need to know all of the factors of the function. The graph of a function can also give you insight into the zeros of a function and their multiplicities.*

$f(x) = \frac{3}{4}(x+1)(x-2)^2$;

— 1 and 1 are zeros of f.

— 1 is a zero of multiplicity 1.

2 is a zero of f with multiplicity 2.

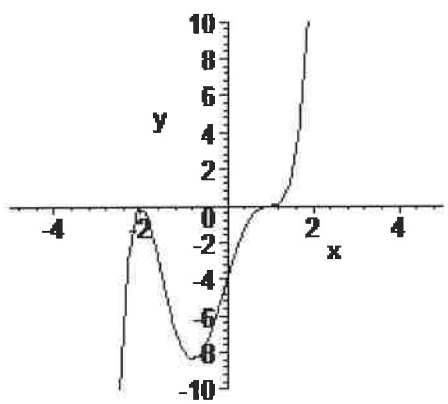


$f(x) = (x+2)^2(x-1)^3$;

— 2 and 1 are zeros of f.

— 2 is a zero of multiplicity 2.

1 is a zero of f with multiplicity 3.



Geometric Meaning of Multiplicity:

If c is a zero of multiplicity n then:

1) When n is odd the graph of the polynomial will cross the x-axis at $(c, 0)$.

2) When n is even the graph of the polynomial will touch the x-axis at $(c, 0)$ but will not pass through.

3) When $n \geq 2$ then the graph of f will "flatten out" as it approaches $(c, 0)$

If $g(x) = 2(x-1)(x+1)^2(x+2)^3$

Determine the degree of g. 6

Determine the y-intercept. $(0, -16)$

Determine the x-intercepts. $(1, 0)$, $(-1, 0)$, $(-2, 0)$

Where does the graph of g cross the x-axis? At $x = 1$, $x = -2$

Where does the graph of g "bounce off" of the x-axis? At $x = -1$

For which $x$intercepts does the graph of g "flatten out" ? At $x = -1$, $x = -2$

Use this information and the end-behaviors to draw a rough sketch of the graph of g .

2
1
0
-1
-2
-3
-4
-3
-2
-1
1
2
3

Completely factor the polynomial function
$P(x) = x^4 - x^3 + 2x^2 - 4x - 8$ and find all of the zeros of P.

$P(-1) = 1 + 1 + 2 + 4 - 8 = 0$

$$
\begin{array}{r l}
 & x^3 - 2x^2 + 4x - 8 \\
x+1 & \overline{)\, x^4 - x^3 + 2x^2 - 4x - 8} \\
 & \underline{x^4 + x^3} \\
 & -2x^3 + 2x^2 - 4x - 8 \\
 & \underline{-2x^3 - 2x^2} \\
 & 4x^2 - 4x - 8 \\
 & \underline{4x^2 + 4x} \\
 & -8x - 8 \\
 & \underline{-8x - 8} \\
 & 0
\end{array}
$$

$$
\begin{aligned}
P(x) &= (x+1)(x^3 - 2x^2 - 4x - 8) \\
&= (x+1)\left[x^2(x-2) - 4(x-2)\right] \\
&= (x+1)(x-2)(x^2-4) \\
&= (x+1)(x-2)(x+2)(x-2) \\
&= (x+1)(x-2)^2(x+2)
\end{aligned}
$$

$P(x) = 0$ when $x = -1$, $x = 2$, $x = -2$

Theorem:
If the imaginary number $a + bi$ is a zero of a polynomial then its conjugate $a - bi$ is also a zero.

If $5 - 3i$ is a zero of the polynomial function $f(x) = x^4 - 6x^3 - 11x^2 + 186x - 170$, find all of the zeros of f and write f in factored form.

$$[x-(5-3i)][x-(5+3i)]$$

$$[(x-5)+3i][(x-5)-3i]$$

$$= (x-5)^2 - 9i^2 = (x-5)^2+9 = x^2-10x+34$$

$$\begin{array}{r|l}
 & x^2+4x-5 \\
x^2-10x+34 & x^4-6x^3-11x^2+186x-170 \\
 & \underline{x^4-10x^3+34x^2} \\
 & 4x^3-45x^2+186x-170 \\
 & \underline{4x^3-40x^2+136x-170} \\
 & -5x^2+50x-170 \\
 & \underline{-5x^2+50x-170} \\
 & 0
\end{array}$$

$$f(x) = (x^2-10x+34)(x^2+4x-5)$$

$$= (x^2-10x+34)(x+5)(x-1)$$

zeros: $5-3i,\ 5+3i,\ -5,\ +1$

Construct a 5th degree polynomial with a leading coefficient of 4 that has 2 as a zero with multiplicity of 2 and -3 is the only other zero.

$$p(x) = 4(x-2)^2(x+3)^3$$