Inverses of Relations and Functions

In many problems, we want to simplify an expression or a relation by "undoing" an operation.

Example: Solve the equation $5\sqrt{x+3} = 10$

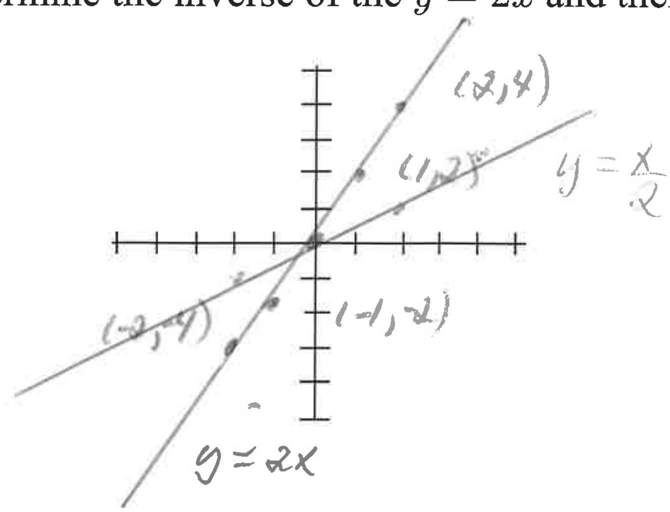
$25(x+3) = 100, \quad x+3 = 4$

$x = 1$

In this section we study inverses of functions and in many contexts, the process of "undoing" is a matter of applying the inverse of a function.

If R is the relation given by
$R : \{(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots (a_n, b_n) \ldots\}$
Then the **inverse of *R***, denoted R^{-1}, is given by
$R^{-1} : \{(b_1, a_1), (b_2, a_2), (b_3, a_3), \ldots (b_n, a_n) \ldots\}$

Example: Determine the inverse of the $y = 2x$ and then graph it on the axis below.



(0,0)
(2,1)
(4,2)
(-2,-1)
(-4,-2)

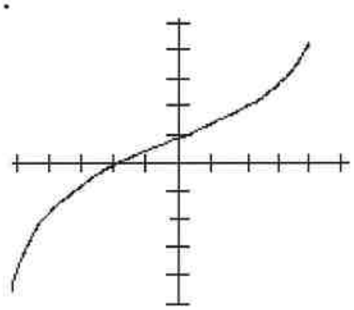
Definition:
A function f is said to be one-to-one if for all $a \neq b$ then $f(a) \neq f(b)$

In other words, a function is one-to-one if there are no output values that are used more than once for a given function.

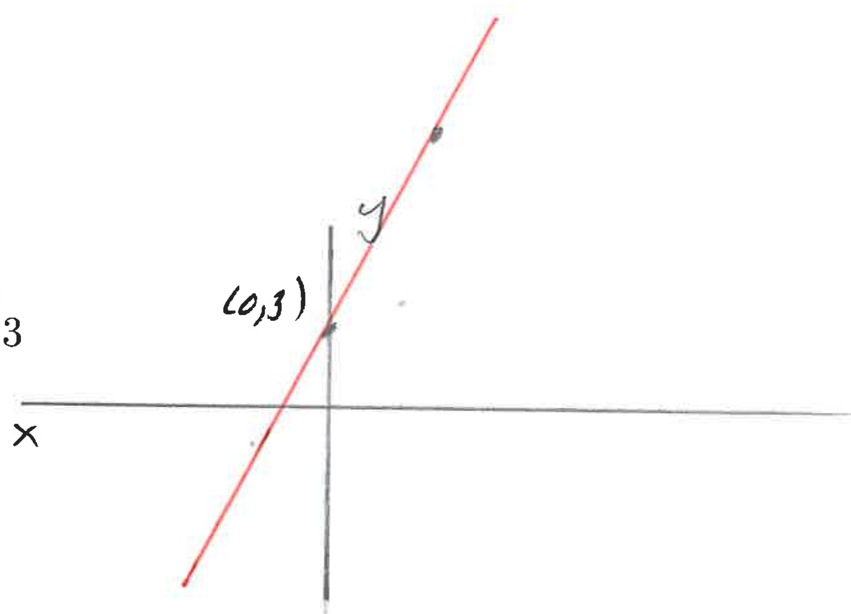
Examples of one-to-one functions:

f : $\{(2,1)\ (3,4)\ (7,12)\ (9,8)\}$

g :



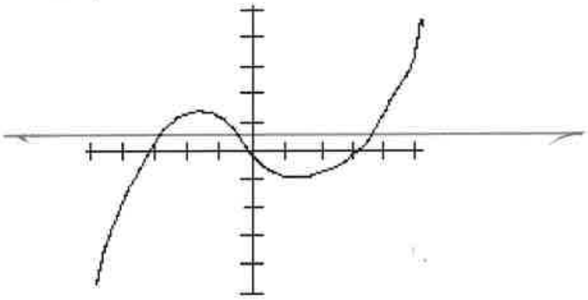
h : $h(x) = 2x + 3$



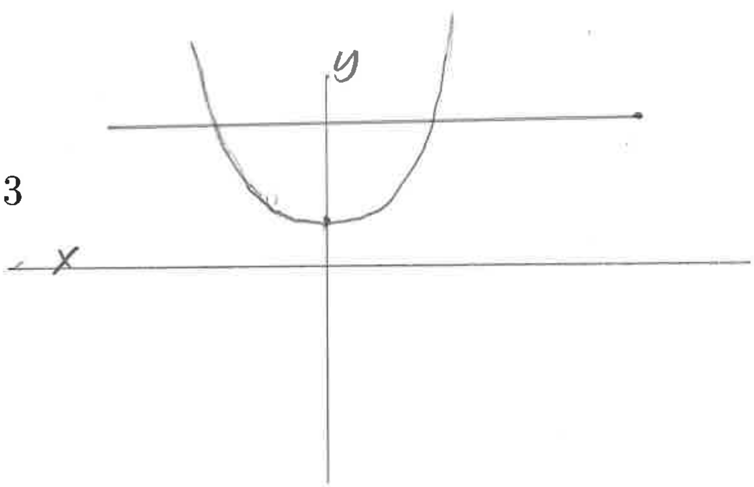
Examples of functions that are not one-to-one:

$F: \{(2,1)\ (3,4)\ (7,12)\ (9,4)\}$

$G:$



$H: \ H(x) = x^2 + 3$



The Horizontal Line Test:
A function is one-to-one if and only if any horizontal line would intersect the graph in at most one place.

The inverse f^{-1} of a function f is also a function if and only if f is one-to-one.

How to find an inverse of a function that is defined as an algebraic equation.

1. If necessary, replace $f(x)$ with y.
2. Switch all x's and y's
3. Solve for y.
4. Replace y with $f^{-1}(x)$

Example:
Find the inverse of $h(x) = \sqrt[3]{\frac{x}{2}}$ $\quad y = \sqrt[3]{\frac{x}{2}}$

$x = \sqrt[3]{\frac{y}{2}}$, $x^3 = \frac{y}{2}$

$y = 2x^3$, $f^{-1}(x) = 2x^3$

Example:
Find the inverse of the function $f(x) = \frac{3x+2}{x+1}$

$y = \frac{3x+2}{x+1}$, $x = \frac{3y+2}{y+1}$, solve for y

$xy + x = 3y + 2$

$xy - 3y = -x + 2$

$y(x-3) = -x+2$, $y = \frac{2-x}{x-3}$

$f^{-1}(x) = \frac{2-x}{x-3}$

Theorem:
The functions f and g are inverse functions if and only if
$(f \circ g)(x) = x$
and
$(g \circ f)(x) = x$

Example:
Show that the function $f(x) = \frac{x^3}{3}$ and $g(x) = \sqrt[3]{3x}$ are inverses

$$(f \circ g)(x) = f[g(x)] = f[\sqrt[3]{3x}]$$

$$= \frac{(\sqrt[3]{3x})^3}{3} = \frac{3x}{3} = x$$

$$(g \circ f)(x) = g[f(x)] = g\left[\frac{x^3}{3}\right]$$

$$= \sqrt[3]{3 \cdot \frac{x^3}{3}} = \sqrt[3]{x^3} = x.$$

$f(x) = \frac{x^3}{3}$, $g(x) = \sqrt[3]{3x}$