Combining Functions

Just like we can add, subtract, multiply, and divide numbers to create new numbers, we can do the same for functions. We can create new functions by combining functions using these arithmetic operations.

Given functions f and g, we can create new functions : The sum $f+g$, difference $f-g$, product fg, and quotient $\frac{f}{g}$. These functions are defined as follows:

Sum $f+g$
$(f+g)(x) = f(x) + g(x)$
where x is in the domain of both f and g.

Difference $f-g$
$(f-g)(x) = f(x) - g(x)$
where x is in the domain of both f and g.

Product fg
$(fg)(x) = f(x)g(x)$
where x is in the domain of both f and g.

Quotient $\frac{f}{g}$
$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$
where x is in the domain of both f and g, and $g(x) \neq 0$

**Notice that the domain for all of these combinations are restricted to values of x that belong to both of the functions that have been combined.*

Example:

Let f be the function $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

Let g be the function $\{(1,0),(2,3),(3,4),(4,6),(5,9),(6,11)\}$

$(f+g)(2) \quad = f(2)+g(2) = 4+3 = 7$

$(f-g)(5) \quad = f(5)-g(5) = 10-9 = 1$

$(f\,g)(4) \quad = f(4)\,g(4) = (8)(6) = 48$

$\left(\frac{f}{g}\right)(3) \quad = \frac{f(3)}{g(3)} = \frac{6}{4} = \frac{3}{2}$

$\left(\frac{f}{g}\right)(1) \quad = \frac{f(1)}{g(1)} = \frac{2}{0}$

Not defined!

Let $f(x) = 2x + 1$ and let $g(x) = \sqrt{x+2}$.
Evaluate the following:

$(f+g)(2) \quad = f(2) + g(2) = 5 + 2 = 7$

$(f+g)(x) \quad = f(x) + g(x) = 2x+1 + \sqrt{x+2}$

$(f \cdot g)(2) \quad = f(2) \cdot g(2) = (5)(2) = 10$

$(f \cdot g)(x) \quad = f(x) \cdot g(x) = (2x+1)(\sqrt{x+2})$

$\frac{f}{g}(2) \quad = \frac{f(2)}{g(2)} = \frac{5}{2} = 2\frac{1}{2}$

$\frac{f}{g}(x) \quad = \frac{f(x)}{g(x)} = \frac{2x+1}{\sqrt{x+2}}$, for $x > -2$

Definition:
Given a function f and a function g, the function
***f composed with g* "*f* ∘ *g*"**, is given by: $(f \circ g)(x) = f(g(x))$.
The domain of $f \circ g$ is all real numbers x that are in the domain of g such that $g(x)$ is in the domain of f.

Example: Let $f(x) = \frac{1}{x+3}$ and let $g(x) = \frac{1}{x}$
Evaluate the following:

$(f \circ g)(2) = f[g(2)] = f[\frac{1}{2}] = \frac{1}{\frac{1}{2}+3} = \frac{2}{7}$

$(f \circ g)(x) = f[g(x)] = f[\frac{1}{x}] = \frac{1}{\frac{1}{x}+3} = \frac{1}{\frac{3x+1}{x}}$

$= \frac{x}{3x+1}$

$(g \circ f)(5) = g[f(5)] = g[\frac{1}{8}] = \frac{1}{\frac{1}{8}} = 8$

$(g \circ f)(x) = g[f(x)] = g[\frac{1}{x+3}] = \frac{1}{\frac{1}{x+3}}$

$= x+3$

What is the domain of $f \circ g$? $x \neq -3,\ x \neq 0$

and $x \neq \frac{-1}{3}$

What is the domain of $g \circ f$?

$x \neq -3,\ x \neq 0$

Example:

Let $g(t) = \frac{3\sqrt{t}}{t+1}$ and let $h(t) = t^2$

find $(g \circ h)(2)$ $= g[h(2)] = g[4] = \frac{6}{5}$

find $(h \circ g)(2)$ $= h[g(2)] = h[\sqrt{2}] = 2$

find $(g \circ h)(t)$ $= g[h(t)] = g[t^2]$

$= \frac{3\sqrt{t^2}}{t^2+1} = \frac{3t}{t^2+1}$, $\underbrace{t \geq 0}_{\text{domain}}$

find $(h \circ g)(t)$ $= h[g(t)] = h\left[\frac{3\sqrt{t}}{t+1}\right]$

$= \left(\frac{3\sqrt{t}}{t+1}\right)^2 = \frac{9t}{t^2+2t+1}$, $\underbrace{t \geq 0}_{\text{domain}}$

Example:
Let f be the function $\{(3,4)\ (4,5)\ (5,7)\ (9,3)\}$
Let g be the function $\{(1,3)\ (2,9)\ (3,7)\ (4,3)\}$

Find the following:

$(f \circ g)(1) =$ $f[g(1)] = f[3] = 4$

$(f \circ g)(2) =$ $f[g(2)] = f[9] = 3$

$(f \circ g)(3) =$ $f[g(3)] = f[7] =$ Not defined

Decompose the function $f(x) = 2(x+1)^2 + 2$ into a composition of two functions.

Let $g(x) = x+1$, $h(x) = 2x^2 + 2$

$f(x) = h[g(x)] = (h \circ g)(x)$

Decompose the function $g(x) = \frac{3}{2\sqrt{3x^2+1}}$ into a composition of two functions.

Let $w(x) = 3x^2 + 1$, $h(x) = \frac{3}{2\sqrt{x}}$

$g(x) = h[w(x)] = (h \circ w)(x)$