Problem Definition

Problem 33. **Maximum Volume:** A rectangular package to be sent by a postal service can have a maximum combined length and girth of 108 inches. Find the dimensions of the package that contains a maximum volume. Assume the dimensions are x by x by y. The girth is the distance around the package perpendicular to the length.

Solution Step 1:

The first step is define the variables used to measure the dimensions. Let's use the following system.

Variable for square end of the package	x
Variable for the length of the package	w

The formula for the volume is given by

$$V = xxw = x^2w$$

and the formula for the dimension restrictions is the following.

$$\text{Girth} + \text{Length} \ = 4x + w = 108$$

Solution Step 2:

The next step is to solve for width w in terms of the other dimension x as follows.

$$w = 108 - 4w$$

This can be substituted into the volume formula. So, the formula for the volume is

$$V = V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$$

Note that the dimensions are positive and the domain for the volume function is the interval $[0, 27]$.

Solution Step 3:

Now we can determine the critical points for the function and then determine the dimensions that maximizes the volume. The derivative of the volume function is

$$\frac{dV}{dx} = 216x - 12x^2 = 12x(18 - x)$$

The critical points as $x = 0$ and $x = 18$

Solution Step 4:

For this problem, we can compute the volume for the critical points and the end points of the interval that is the domain of the volume function. This gives

$$\begin{array}{lll} x = 0,\ w = 108 & \mathrm{V(0)=0} & \\ x = 18,\ w = 36 & \mathrm{V(18)=11664} & \text{absolute maximum} \\ x = 27,\ w = 0 & \mathrm{V(27)=0} & \end{array}$$

As another test we could have used the first or second derivative tests to show that $x = 18$ is a relative maximum.

Solution Step 5:

The maximum volume of 11664 cubic inches is obtained for the dimensions $x = 18$ and $w = 36$.